Physics 498/MMA **Mathematical Methods in Physics I** Prof. M. Stone
Handout 7 `http://w3.physics.uiuc.edu/~m-stone5` 2117 ESB
Fall 2007 HOMEWORK 7 University of Illinois

1) Critical Mass: An infinite slab of fissile material has thickness L. The neutron density $n(\mathbf{r})$ in the material obeys the equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n + \lambda n + \mu,$$

where n is zero at the surface of the slab at $x = 0, L$. Here D is the neutron diffusion constant, the term λn describes the creation of new neutrons by induced fission, and μ is the rate of production per unit volume of neutrons by spontaneous fission. Assume that n depends only on x.

a) Expand n as a series
$$n(x,t) = \sum_m a_m(t)\varphi_m(x)$$
where the φ_m are a complete set of functions you think suitable for solving the problem.

b) Find an explicit expression for the coefficients $a_m(t)$ in terms of their intial values $a_m(0)$.

c) Determine the critical thickness, L_{crit}, above which the slab will explode.

d) Assuming that $L < L_{\text{crit}}$, find the equilibrium distribution $n_{\text{eq}}(x)$ of neutrons in the slab. (You may either sum your series expansion to get an explicit closed-form answer, or use another (Green function?) method.)

2) Semi-infinite Rod: Consider the heat equation

$$\frac{\partial \theta}{\partial t} = D\nabla^2\theta, \quad 0 < x < \infty$$

with the temperature $\theta(x,t)$ obeying the initial condition $\theta(x,0) = \theta_0$ for $0 < x < \infty$, and the boundary condition $\theta(0,t) = 0$.

a) Show that the boundary condition at $x = 0$ can be satisfied at all times by introducing a suitable mirror image of the initial data in the region $-\infty < x < 0$, and then applying the heat kernel for the entire real line to this extended initial data. Show that the solution of the semi-infinite rod problem can be expressed in terms of the *error function*
$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\,d\xi.$$

b) Solve the same problem by using a Fourier integral expansion in terms of $\sin kx$ on the half-line $0 < x < \infty$ and obtaining the time evolution of the Fourier coefficients. Invert the transform and show that your answer reduces to that of part a). (Hint: replace the initial condition by $\theta(x,0) = \theta_0 e^{-\epsilon x}$, so that the Fourier transform converges, and then take the limit $\epsilon \to 0$ at the end of your calculation.)

3) Jacobi's Imaginary Transformation: Consider the heat equation

$$\frac{\partial^2 \phi}{\partial \theta^2} = \frac{\partial \phi}{\partial t}$$

on a circle of circumference 2π.

a) Use the method of images to show that the Green function can be written as

$$G(\theta, 0; t, 0) = \sqrt{\frac{1}{4\pi t}} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{1}{4t}(\theta + 2\pi n)^2\right\}.$$

b) Hence or otherwise establish the identity

$$\sqrt{\frac{1}{2\pi t}} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{1}{2t}(\theta + 2\pi n)^2\right\} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{1}{2}tn^2 + in\theta\right\}.$$

Observe that, depending on whether t is large or small, either the left or right hand side converges rapidly while the sum on the other side of the equation converges slowly.

This result is an example of the *Poisson summation* formula

$$\sum_{n=-\infty}^{\infty} F(n) = \sum_{n=-\infty}^{\infty} \tilde{F}(2\pi n)$$

where $\tilde{F}(k) = \int_{-\infty}^{\infty} e^{ikx} F(x)\, dx$.

4) Seasonal Heat Waves: Suppose that the measured temperature of the air above the arctic permafrost is expressed as a Fourier series

$$\theta(t) = \theta_0 + \sum_{n=1}^{\infty} \theta_n \cos n\omega t,$$

where $T = 2\pi/\omega$ is one year. Solve the heat equation for the soil temperature

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial z^2}, \quad 0 < z < \infty$$

with this boundary condition, and find the temperature $\theta(z, t)$ at a depth z below the surface as a function of time. Observe that the sub-surface temperature fluctuates with the same period as that of the air, but with a phase lag that depends on the depth. Also observe that the longest period temperature fluctuations penetrate the deepest into the ground. (Hint: for each Fourier component, write θ as $\mathrm{Re}[A_n(z) \exp in\omega t]$ where A_n is a complex function of z.)